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# Simplicial Maps from an Orientable $n$-Pseudomanifold into $S^m$ with the Octahedral Triangulation*

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ABSTRACT

A combinatorial theorem on simplicial maps from an orientable $n$-pseudomanifold into an $m$-sphere with the octahedral triangulation is proved. Special results unified by this theorem are discussed.

## 1. DEFINITIONS AND NOTATION

For convenience of the reader, we recall that an *n-dimensional pseudomanifold* or an *n-pseudomanifold* is a finite simplicial complex $M^n$ satisfying the following three conditions:

(a) Every simplex (of any dimension) of $M^n$ is a face of at least one $n$-simplex of $M^n$.

(b) Every $(n-1)$-simplex of $M^n$ is a face of at most two $n$-simplexes of $M^n$.

(c) For any two $n$-simplexes $\sigma$, $\tau$ of $M^n$, there is a finite sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k = \tau$ of $n$-simplexes of $M^n$ such that for each $i = 1, 2, \ldots, k-1$, $\sigma_i$ and $\sigma_{i+1}$ have an $(n-1)$-face in common.

An $(n-1)$-simplex $\sigma$ of an $n$-pseudomanifold $M^n$ is called a *boundary $(n-1)$-simplex* of $M^n$, if $\sigma$ is a face of exactly one $n$-simplex of $M^n$.

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An $n$-pseudomanifold $M^n$ is said to be *without boundary*, if there is no boundary $(n - 1)$-simplex. Two vertices of $M^n$ are said to be *adjacent*, if they are the two vertices of a 1-simplex of $M^n$.

For an oriented $n$-simplex $\sigma$ with vertices $v_0, v_1, ..., v_n$, we shall write $\sigma = + v_0 v_1 ... v_n$ if the orientation of $\sigma$ is that given by the even permutations of the arrangement $(v_0, v_1, ..., v_n)$. We shall write $\sigma = - v_0 v_1 ... v_n$ if the orientation of $\sigma$ is that given by the odd permutations of the arrangement $(v_0, v_1, ..., v_n)$.

An $n$-pseudomanifold $M^n$ is *coherently oriented*, if all its $n$-simplexes and all its boundary $(n - 1)$-simplexes are so oriented that:

(i) whenever an $(n - 1)$-simplex $\tau$ is a common face of two $n$-simplexes $\sigma_1$ and $\sigma_2$, the orientations of $\sigma_1$ and $\sigma_2$ induce opposite orientations on $\tau$;

(ii) each boundary $(n - 1)$-simplex has the orientation induced by the orientation of the incident $n$-simplex.

An $n$-pseudomanifold is said to be *orientable* if it can be coherently oriented.

In the following we shall consider a coherently oriented $n$-pseudomanifold $M^n$ and a function $\varphi$ which assigns to each vertex $v$ of $M^n$ an integer $\varphi(v)$. For $n + 1$ distinct integers $j_0, j_1, ..., j_n$ arranged in this order, we shall denote by $\alpha_+(j_0, j_1, ..., j_n)$ the number of those $n$-simplexes $\sigma$ of $M^n$ such that $\sigma = + v_0 v_1 ... v_n$ and $\varphi(v_i) = j_i (0 \leq i \leq n)$. $\alpha_-(j_0, j_1, ..., j_n)$ will denote the number of those $n$-simplexes $\sigma$ of $M^n$ such that $\sigma = - v_0 v_1 ... v_n$ and $\varphi(v_i) = j_i (0 \leq i \leq n)$. We define

$$\alpha(j_0, j_1, ..., j_n) = \alpha_+(j_0, j_1, ..., j_n) - \alpha_-(j_0, j_1, ..., j_n). \quad (1)$$

Similarly, for any $n$ distinct integers $j_0, j_1, ..., j_{n-1}$ arranged in this order, $\beta_+(j_0, j_1, ..., j_{n-1})$ will denote the number of those boundary $(n - 1)$-simplexes $\sigma$ of $M^n$ such that $\sigma = + v_0 v_1 ... v_{n-1}$ and $\varphi(v_i) = j_i$ $(0 \leq i \leq n - 1)$, while $\beta_-(j_0, j_1, ..., j_{n-1})$ will denote the number of those boundary $(n - 1)$-simplexes $\sigma$ of $M^n$ such that $\sigma = - v_0 v_1 ... v_{n-1}$ and $\varphi(v_i) = j_i (0 \leq i \leq n - 1)$. We define

$$\beta(j_0, j_1, ..., j_{n-1}) = \beta_+(j_0, j_1, ..., j_{n-1}) - \beta_-(j_0, j_1, ..., j_{n-1}). \quad (2)$$

In case $M^n$ is without boundary, we have of course

$$\beta(j_0, j_1, ..., j_{n-1}) = 0.$$

## 2. A Combinatorial Theorem

Let $S^m$ denote the $m$-sphere in the Euclidean $(m + 1)$-space $R^{m+1}$ formed by all points

$$x = (x_1, x_2, ..., x_{m+1})$$

with coordinates satisfying $\sum_{i=1}^{m+1} x_i^2 = 1$. By *octahedral triangulation* of $S^m$ we understand the subdivision of $S^m$ into $2^{m+1}$ spherical $m$-simplexes by means of the $m + 1$ coordinate hyperplanes of $R^{m+1}$. The function $\varphi$ considered in the following theorem can be regarded as a simplicial map from $M^n$ into $S^m$ with the octahedral triangulation, provided $m + 1 \geq \max | \varphi(v) |$.

THEOREM 1. *Let $M^n$ be a coherently oriented $n$-pseudomanifold. Let $\varphi$ be a function defined on the set of vertices of $M^n$ such that:*

*For each vertex $v$ of $M^n$, $\varphi(v)$ is a non-zero integer, positive or negative.* (3)

$\varphi(u) + \varphi(v) \neq 0$ *for any two adjacent vertices $u$, $v$ of $M^n$.* (4)

*Then we have*

$$\sum_{0<k_0<k_1<\cdots<k_n} \{\alpha(-k_0, k_1, -k_2, k_3, ..., (-1)^{n+1}k_n) + (-1)^n\alpha(k_0, -k_1, k_2, -k_3, ..., (-1)^n k_n)\} = \sum_{0<k_0<k_1<\cdots<k_{n-1}} \beta(k_0, -k_1, k_2, -k_3, ..., (-1)^{n-1}k_{n-1}). \tag{5}$$

*In particular, if $M^n$ is without boundary, we have*

$$\sum_{0<k_0<k_1<\cdots<k_n} \alpha(-k_0, k_1, -k_2, k_3, ..., (-1)^{n+1}k_n) = (-1)^{n+1} \sum_{0<k_0<k_1<\cdots<k_n} \alpha(k_0, -k_1, k_2, -k_3, ..., (-1)^n k_n). \tag{6}$$

EXAMPLE 1. Let $M^2$ be the annular ring with the triangulation given in Figure 1. $M^2$ is so coherently oriented that the orientation for each 2-simplex is chosen to be the counterclockwise orientation. The values of $\varphi$ are indicated by the integers marked at the vertices. There are five 2-simplexes which contribute to the left side of (5). Two of them are marked with a counterclockwise arrow, the three others are marked with a clockwise arrow. Each of these circular arrows indicates that orientation of a 2-simplex which corresponds to even permutations of the increasing order of the absolute values of the integers at the vertices. Thus the 2-simplex with integers $-2, 5, -23$ at its vertices is marked with a clockwise arrow, so this 2-simplex contributes to $\alpha_-(-2,5,-23) = 1$. Similarly, the 2-simplex with integers $-2, 9, -23$ at its vertices

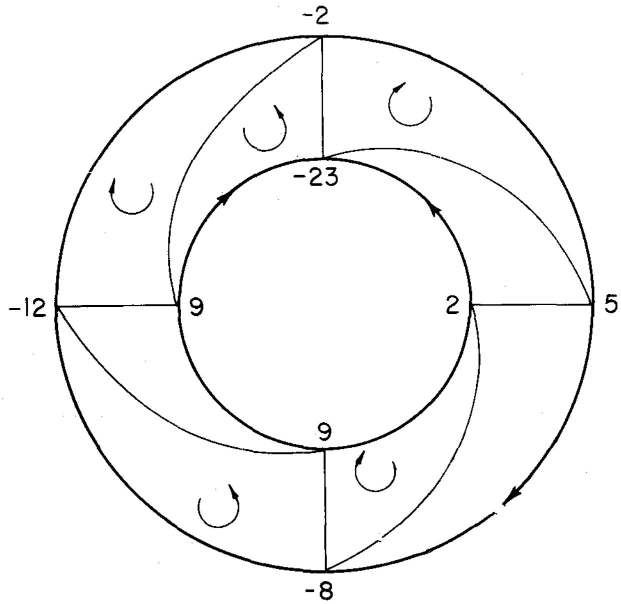


FIGURE 1

is marked with a counterclockwise arrow and contributes to $\alpha_+(-2,9, -23) = 1$. We count

$$\begin{aligned}\sum_{0<k_0<k_1<k_2} \alpha(-k_0, k_1, -k_2) &= \alpha_+(-2,9,-23) + \alpha_+(-8,9,-12) \\ &\quad - \alpha_-(-2,5,-23) - \alpha_-(-2,9,-12) \\ &= 1 + 1 - 1 - 1 = 0, \\ \sum_{0<k_0<k_1<k_2} \alpha(k_0, -k_1, k_2) &= -\alpha_-(2,-8,9) = -1,\end{aligned}$$

so

$$\sum_{0<k_0<k_1<k_2} \{\alpha(-k_0, k_1, -k_2) + (-1)^2\alpha(k_0, -k_1, k_2)\} = -1.$$

There are three boundary 1-simplexes which contribute to the right side of (5). Each of them is also marked with an arrow indicating the increasing order of the absolute values of the integers at the vertices. The arrow on the boundary 1-simplex with integers 9, $-23$ at its vertices agrees with the orientation in the chosen coherent orientation of $M^2$, so this boundary 1-simplex contributes to $\beta_+(9, -23) = 1$. The other two boundary 1-simplexes marked with arrow are counted for $\beta_-$. We have

$$\sum_{0<k_0<k_1} \beta(k_0, -k_1) = \beta_+(9, -23) - \beta_-(2, -23) - \beta_-(5, -8) = 1 - 1 - 1 = -1.$$

Thus in this example, both sides of (5) are equal to $-1$.

EXAMPLE 2. Figure 2 is a triangulation of a torus. The opposite sides of the large square are to be identified. Let each 2-simplex be oriented counterclockwise. The values of $\varphi$ are marked at each vertex. (Observe

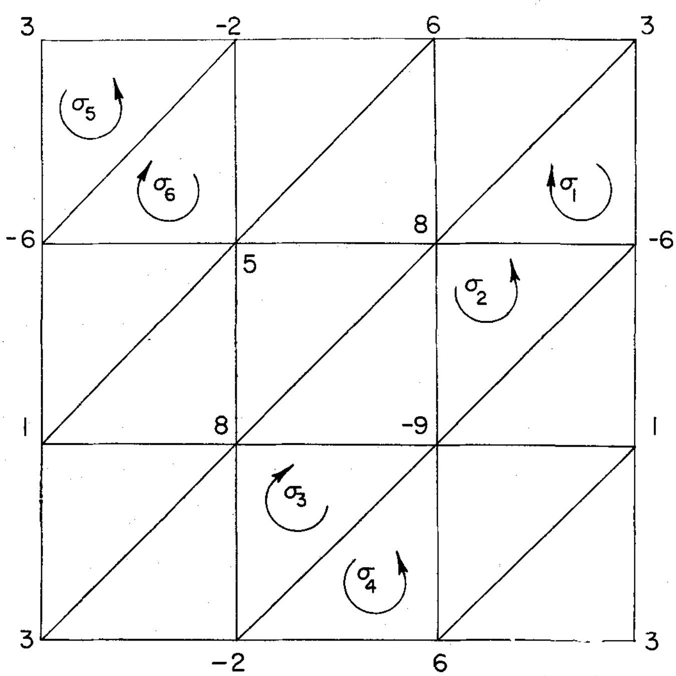


FIGURE 2

that the values of $\varphi$ at any two vertices representing same point on the torus are same.) $\sigma_1$ is the only 2-simplex which contributes to the right side of (6). $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$ and $\sigma_6$ are the only 2-simplexes which contribute to the left side of (6). Both sides of (6) are equal to 1.

## 3. Proof of Theorem 1

We begin with a simple observation which will be crucial in our proof. Suppose the $n$-simplexes of $M^n$ are distributed into two disjoint non-empty classes. Let $M_1{}^n$, $M_2{}^n$ be the simplicial $n$-complexes formed by all $n$-simplexes of the first class and of the second class, respectively, together with their faces. Then both $M_1{}^n$ and $M_2{}^n$ have properties (a), (b) of $n$-pseudomanifolds, although condition (c) may not be fulfilled. Let the $n$-simplexes of $M_1{}^n$, $M_2{}^n$ be oriented as in $M^n$. Let each boundary $(n-1)$-simplex of $M_1{}^n$ (or $M_2{}^n$) be given the orientation induced by that of the incident $n$-simplex in $M_1{}^n$ (or $M_2{}^n$). Let $\varphi_1$ and $\varphi_2$ be the restrictions of $\varphi$ on the set of vertices of $M_1{}^n$ and $M_2{}^n$, respectively. If relation (5) holds for both $\varphi_1$ on $M_1{}^n$ and $\varphi_2$ on $M_2{}^n$, then, since $M^n$ is coherently oriented, (5) also holds for $M^n$. We may express this fact by saying that relation (5) is additive.

Because of this additive property of (5), it suffices to consider the case in which $M^n$ consists of a single $n$-simplex and its faces. From now on, we shall assume that $M^n$ is formed by a single $n$-simplex $\sigma$ and its faces, where $\sigma$ is oriented and each of its $(n-1)$-faces has the orientation induced from that of $\sigma$. Then the left side of (5) is equal to 1 or $-1$ or 0. We have to discuss separately the following cases.

Case 1. The left side of (5) is equal to $\varepsilon = \pm 1$ and

$$\alpha(-j_0, j_1, -j_2, \ldots, (-1)^{n+1}j_n) = \varepsilon$$

for certain integers $0 < j_0 < j_1 < j_2 < \cdots < j_n$.

In this case, let the vertices of $\sigma$ be denoted by $v_0, v_1, \ldots, v_n$ in such a way that

$$\varphi(v_i) = (-1)^{i+1}j_i \quad (0 \leq i \leq n).$$

Then we must have $\sigma = \varepsilon v_0 v_1 \ldots v_n$. Furthermore, the oriented $(n-1)$-face $\varepsilon v_1 v_2 \ldots v_n$ of $\sigma$ is the only one which yields the only non-zero term

$$\beta(j_1, -j_2, \dots, (-1)^{n+1}j_n) = \varepsilon$$

on the right side of (5). Thus both sides of (5) are equal to $\varepsilon$.

CASE 2. The left side of (5) is equal to $\varepsilon = \pm 1$ and

$$(-1)^n \alpha(j_0, -j_1, j_2, \dots, (-1)^n j_n) = \varepsilon$$

for certain integers $0 < j_0 < j_1 < \dots < j_n$.

In this case, let the vertices of $\sigma$ be denoted by $v_0, v_1, \dots, v_n$ such that

$$\varphi(v_i) = (-1)^i j_i \quad (0 \leq i \leq n).$$

Then we must have $\sigma = (-1)^n \varepsilon v_0 v_1 \dots v_n$. Furthermore the oriented $(n-1)$-face $\varepsilon v_0 v_1 \dots v_{n-1}$ of $\sigma$ is the only one which gives the only non-zero term

$$\beta(j_0, -j_1, j_2, \dots, (-1)^{n-1} j_{n-1}) = \varepsilon$$

on the right side of (5). Thus both sides of (5) are equal to $\varepsilon$.

CASE 3. The left side of (5) vanishes.

In this case, we show that the right side of (5) also vanishes. Clearly it suffices to consider the case where the values of $\varphi$ at $n$ of the vertices of $\sigma$ are

$$j_0, -j_1, j_2, \dots, (-1)^{n-1} j_{n-1}$$

with certain integers $0 < j_0 < j_1 < \cdots < j_{n-1}$. Denote by $h$ the value of $\varphi$ at the remaining vertex of $\sigma$. We discuss two subcases separately.

CASE 3.1. $|h| = j_i$ for some $i$ between 0 and $n-1$.

In this case we must have $h = (-1)^i j_i$ because of hypothesis (4). Let the vertices of $\sigma$ be denoted by $v_0, v_1, \dots, v_n$ such that $\sigma = \varepsilon v_0 v_1 \dots v_n$ with $\varepsilon = \pm 1$ and

$$\begin{aligned} \varphi(v_0) = j_0, \varphi(v_1) = -j_1, \dots, \varphi(v_{i-1}) &= (-1)^{i-1} j_{i-1}, \\ \varphi(v_i) = \varphi(v_{i+1}) &= (-1)^i j_i, \\ \varphi(v_{i+2}) = (-1)^{i+1} j_{i+1}, \dots, \varphi(v_n) &= (-1)^{n-1} j_{n-1}. \end{aligned}$$

Then corresponding to the two oriented $(n-1)$-faces

$$(-1)^i \varepsilon v_0 v_1 \dots v_{i-1} v_{i+1} \dots v_n \qquad \text{and} \qquad (-1)^{i+1} \varepsilon v_0 v_1 \dots v_i v_{i+2} \dots v_n$$

of $\sigma$ we have

$$\beta(j_0, -j_1, j_2, \dots, (-1)^{n-1}j_{n-1}) = (-1)^i \varepsilon + (-1)^{i+1} \varepsilon = 0.$$

Therefore the right side of (5) vanishes.

CASE 3.2. $|h|$ is different from each of $j_0, j_1, \dots, j_{n-1}$.

In this case we have either $j_i < |h| < j_{i+1}$ for some $i = 0, 1, \dots, n-2$, or $|h| < j_0$, or $j_{n-1} < |h|$.

Assume first that $j_i < |h| < j_{i+1}$ for some $i = 0, 1, \dots, n-2$. Let the vertices of $\sigma$ be denoted by $v_0, v_1, \dots, v_n$ such that $\sigma = \varepsilon v_0 v_1 \dots v_n$ with $\varepsilon = \pm 1$ and

$$\begin{aligned} \varphi(v_0) &= j_0, \varphi(v_1) = -j_1, \dots, \varphi(v_i) = (-1)^i j_i, \\ \varphi(v_{i+1}) &= h, \\ \varphi(v_{i+2}) &= (-1)^{i+1} j_{i+1}, \dots, \varphi(v_n) = (-1)^{n-1} j_{n-1}. \end{aligned}$$

If $h = (-1)^i |h|$, then corresponding to the two oriented $(n-1)$-faces

$$(-1)^i \varepsilon v_0 v_1 \dots v_{i-1} v_{i+1} \dots v_n \quad \text{and} \quad (-1)^{i+1} \varepsilon v_0 v_1 \dots v_i v_{i+2} \dots v_n$$

of $\sigma$, the right side of (5) has exactly two non-zero terms, namely,

$$\beta(j_0, -j_1, \dots, (-1)^{i-1} j_{i-1}, (-1)^i |h|, (-1)^{i+1} j_{i+1}, \dots, (-1)^{n-1} j_{n-1}) = (-1)^i \varepsilon$$

and

$$\beta(j_0, -j_1, \dots, (-1)^i j_i, (-1)^{i+1} j_{i+1}, \dots, (-1)^{n-1} j_{n-1}) = (-1)^{i+1} \varepsilon.$$

Therefore the right side of (5) is equal to $(-1)^i \varepsilon + (-1)^{i+1} \varepsilon = 0$ if $h = (-1)^i |h|$. If $h = (-1)^{i+1} |h|$, we consider the two oriented $(n-1)$-faces

$$(-1)^{i+1} \varepsilon v_0 v_1 \dots v_i v_{i+2} \dots v_n \quad \text{and} \quad (-1)^{i+2} \varepsilon v_0 v_1 \dots v_{i+1} v_{i+3} \dots v_n$$

of $\sigma$, and we infer that the right side of (5) is again equal to zero.

Similarly it can be seen that the right side of (5) again vanishes if $|h| < j_0$ or if $j_{n-1} < |h|$. It suffices to observe that, because the left side of (5) vanishes (remember that we are in Case 3!), $h$ must be positive if $|h| < j_0$, and $h$ must have the same sign as $(-1)^{n-1} j_{n-1}$ if $j_{n-1} < |h|$. This finishes Case 3.2 and thereby completes the proof of Theorem 1.

## 4. A Pairing Process

The proof given in Section 3 is based on the additive property of (5). The use of additive property can be replaced by an explicit pairing process, which is a refinement of the search algorithm used by Cohen [2] in his proof of Sperner's lemma.

Let us first illustrate our pairing process in a practical situation. Consider a house which naturally has rooms and doors. Some of the doors are *internal*, i.e., doors connecting two rooms. Some of the doors are *external*, i.e., connecting a room with the outside of the house. We make the following assumptions (7)–(10):

Each room has 0 or 1 or 2 doors. (7)

For every internal door, one side is painted white and the other side black. (8)

For every external door, both sides are painted with the same color, white or black. (9)

For any room with two doors, the sides of the two doors facing the interior of the room are painted with different colors, one white and one black. (10)

A room will be considered as *good*, if it has exactly one door. If the side of the unique door facing the interior of a good room is white (black), we say that this is a *good room with a white* (*black*) *door*. Under the assumptions (7)–(10), we can conclude that

The number of good rooms with a white door minus the number of good rooms with a black door is equal to the number of white external doors minus the number of black external doors. (11)

Example 3. In Figure 3, the rectangular house has 16 triangular rooms. The internal doors are marked with , where the unbroken line indicates the black side and the broken line indicates the white side. A black external door is marked with , and a white external door with . There are 2 good rooms with a white door, and 2 good rooms with a black door. There are 2 white external doors and 2 black external doors.

That (11) is a consequence of (7)–(10) can be easily seen by consider-

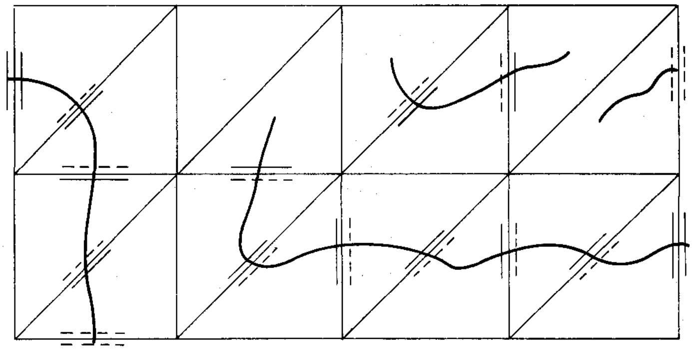

FIGURE 3

ing the associated incidence matrix with entries 0, 1, $-1$. Less simple is the proof described below involving a pairing process, which is, however, an explicit algorithm.

Let us first go through every door exactly once. To start a path, we either enter a room from the outside of the house through an external door, or go out of a good room through its unique door. Everytime when we enter a room through a door and see a second door in the room, we go out of the room through the second door. This will lead us either to the outside of the house or to a new room. According to this rule, our path will be stopped either when we get to the outside of the house through an external door, or when we enter a good room. When we are stopped in either of these two situations, we start a new path. Because of (8), (9), and (10), if both ends of a path are external doors, then these two external doors must be of different colors. For the same reason, if one end of a path is a white (black) external door and the other end is a good room, then this must be a good room with a white (black) door. Also, if both ends of a path are good rooms, then one of them is a good room with a white door, and the other is a good room with a black door. (See the four paths in Figure 3.)

After going through every door exactly once by walking on various paths, we pair off the two ends of each path. Then every white (black) external door is paired either with a black (white) external door or with a good room with a white (black) door. Also each good room with a white (black) door is paired either with a good room with a black (white) door, or with a white (black) external door. Clearly this pairing implies relation (11).

To give an alternative proof of the theorem in Section 2, we translate the theorem to the above practical situation by the following dictionary. The house is $M^n$, the rooms are the $n$-simplexes of $M^n$. The doors are those $(n-1)$-simplexes of $M^n$ such that the values of $\varphi$ at its $n$ vertices are integers

$$k_0, -k_1, k_2, ..., (-1)^{n-1}k_{n-1}$$

satisfying $0 < k_0 < k_1 < \cdots < k_{n-1}$. A door is external if it is a boundary $(n-1)$-simplex of $M^n$. If an $(n-1)$-simplex $\tau$ with vertices $v_i (0 \leq i \leq n-1)$, with

$$\varphi(v_i) = (-1)^i k_i \ (0 \leq i \leq n-1)$$

and $0 < k_0 < k_1 < \cdots < k_{n-1}$, is a face of an $n$-simplex $\sigma$, then the side of this door $\tau$ facing the interior of the room $\sigma$ is painted white (black) if the orientation of $\tau$ induced from that of $\sigma$ (which is given by the chosen coherent orientation of $M^n$) is given by the even (odd) permutations of the arrangement $v_0, v_1, ..., v_{n-1}$. In case this door $\tau$ is an external one, both sides of the door are painted with the same color. Thus conditions (8) and (9) are satisfied.

By an argument similar to that used in Section 3, one can verify (7) and (10). Also one can verify that the good rooms with a white door are precisely those oriented $n$-simplexes $\sigma = + v_0 v_1 ... v_n$ such that

$$\varphi(v_i) = (-1)^{i+1} k_i \qquad (0 \leq i \leq n)$$

with $0 < k_0 < k_1 < \cdots < k_n$, and those oriented $n$-simplexes $\sigma = (-1)^n v_0 v_1 ... v_n$ such that

$$\varphi(v_i) = (-1)^i k_i \qquad (0 \leq i \leq n)$$

with $0 < k_0 < k_1 < \cdots < k_n$. Similarly it can be seen that the good rooms with a black door are precisely those oriented $n$-simplexes $\sigma = - v_0 v_1 ... v_n$ such that

$$\varphi(v_i) = (-1)^{i+1} k_i \qquad (0 \leq i \leq n)$$

with $0 < k_0 < k_1 < \cdots < k_n$, and those oriented $n$-simplexes $\sigma = (-1)^{n+1} v_0 v_1 ... v_n$ such that

$$\varphi(v_i) = (-1)^i k_i \qquad (0 \leq i \leq n)$$

with $0 < k_0 < k_1 < \cdots < k_n$. Hence relation (11) becomes (5).

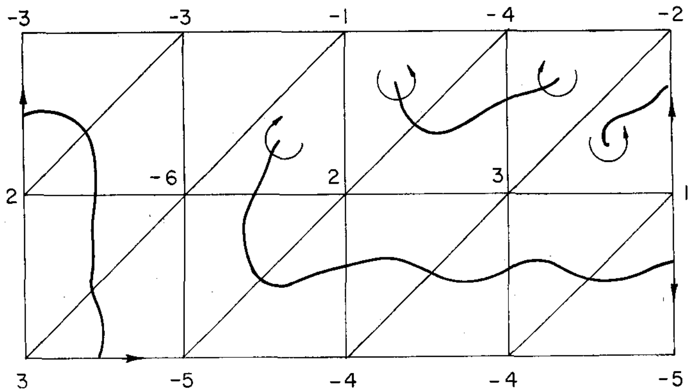


FIGURE 4

EXAMPLE 4. Consider $M^2$ (with the counterclockwise orientation) and $\varphi$ given in Figure 4. Using the dictionary described above, this is translated into the house of Figure 3.

## 5. SPECIAL CASES

In this section we discuss some immediate corollaries of Theorem 1.

COROLLARY 1. *Let $M^n$ be a coherently oriented n-pseudomanifold. Let $\varphi$ be a function which assigns to each vertex of $M^n$ one of the integers* 0, 1, 2, ..., $n + 1$. *Then*

$$\begin{aligned}(-1)^k\alpha(0, 1, ..., k-1, k+1, ..., n+1) \\ -(-1)^h\alpha(0, 1, ..., h-1, h+1, ..., n+1) \\ = (-1)^{h+k}\beta(0, 1, ..., h-1, h+1, ..., k-1, k+1, ..., n+1)\end{aligned} \tag{12}$$

*holds for* $0 \leq h < k \leq n+1$.

PROOF: Consider first the case $h = 0, k = n+1$. In this case, (12) may be written

$$\alpha(0, 1, ..., n) + (-1)^n\alpha(1, 2, ..., n+1) = \beta(1, 2, ..., n). \tag{13}$$

If we replace the integers 0, 1, 2, ..., $k$, ..., $n + 1$, respectively, by

$$-1, 2, -3, ..., (-1)^{k+1}(k+1), ..., (-1)^n(n+2),$$

(13) becomes

$$\begin{aligned}&\alpha(-1, 2, -3, \ldots, (-1)^{n+1}(n+1)) \\ &\quad + (-1)^n \alpha(2, -3, 4, \ldots, (-1)^n(n+2)) \\ &\quad = \beta(2, -3, 4, \ldots, (-1)^{n+1}(n+1)).\end{aligned} \tag{14}$$

By our Theorem 1, (14) holds for any function $\varphi$ taking its values among the integers $-1, 2, -3, 4, \ldots, (-1)^n(n+2)$. Hence the case $h = 0$, $k = n + 1$ of (12) is proved.

To obtain the general case (12), we first exchange 0 with $h$, and $n + 1$ with $k$ in (13); then we rearrange the integers inside each pair of parentheses in increasing order and take care of the changes of orientation affected by the rearrangements.

If, in addition to the hypothesis of Corollary 1, $M^n$ is without boundary, then (12) asserts that the number

$$(-1)^k \alpha(0, 1, \ldots, k-1, k+1, \ldots, n+1)$$

is independent of $k$. This number is, of course, the degree of $\varphi$, which is a simplicial map from the orientable $n$-pseudomanifold $M^n$ without boundary to an $n$-sphere (the boundary of an $(n+1)$-simplex).

COROLLARY 2. *Let $M^n$ be a coherently oriented n-pseudomanifold. Let $\varphi$ be a function which assigns to each vertex of $M^n$ an integer chosen from* $0, 1, 2, \ldots, n$. *Then*

$$\alpha(0, 1, 2, \ldots, n) = (-1)^h \beta(0, 1, \ldots, h-1, h+1, \ldots, n) \tag{15}$$

*holds for* $0 \leq h \leq n$.

PROOF: This is the case $k = n + 1$ of (12). Since $\varphi$ takes its values among $0, 1, 2, \ldots, n$ only, we have

$$\alpha(0, 1, \ldots, h-1, h+1, \ldots, n+1) = 0$$

for $0 \leq h \leq n$.

In case $M^n$ is obtained by triangulating an $n$-simplex, Corollary 2 implies Sperner's lemma with the consideration of orientation. This oriented version of Sperner's lemma is due to Arthur B. Brown [1, p. 133].

Another special case of Theorem 1 is obtained by disregarding orientation. Let $\alpha_0(j_0, j_1, ..., j_n)$ denote the number of those $n$-simplexes (regardless of orientation) in $M^n$ whose $n + 1$ vertices are assigned the integers $j_0, j_1, ..., j_n$, and let $\beta_0(j_0, j_1, ..., j_{n-1})$ denote the number of those boundary $(n - 1)$-simplexes (regardless of orientation) in $M^n$ whose $n$ vertices are assigned the integers $j_0, j_1, ..., j_{n-1}$. Clearly (5) implies the congruence

$$\begin{aligned} \sum_{0<k_0<k_1<\cdots<k_n} & \{\alpha_0(-k_0, k_1, -k_2, k_3, ..., (-1)^{n+1}k_n) \\ & + \alpha_0(k_0, -k_1, k_2, -k_3, ..., (-1)^n k_n)\} \\ \equiv \sum_{0<k_0<k_1<\cdots<k_{n-1}} & \beta_0(k_0, -k_1, k_2, -k_3, ..., (-1)^{n-1}k_{n-1}), \quad \bmod 2. \end{aligned} \tag{16}$$

This weaker result (16) was proved in [4] for any simplicial $n$-complex $M^n$ having properties (a), (b) of the $n$-pseudomanifolds and for any function $\varphi$ satisfying conditions (3), (4).

If $M^n$ is obtained by a further centrally symmetric triangulation of the octahedral triangulation of $S^n$, and if, in addition to (3), (4), $\varphi$ satisfies the antipodal condition that $\varphi(-v) = -\varphi(v)$ for any two antipodal vertices $v$, $-v$ of $M^n$, then by induction on $n$, one can derive

$$\sum_{0<k_0<k_1<\cdots<k_n} \alpha_0(k_0, -k_1, k_2, -k_3, ..., (-1)^n k_n) \equiv 1, \quad \bmod 2 \tag{17}$$

as a direct consequence of (16). This congruence (17) is equivalent to a result of Krasnosel'skii–Krein [5], [6, p. 95]. It was obtained independently in [3] as a generalization of a lemma of Tucker [7], and was used to prove antipodal point theorems generalizing the classical theorems of Borsuk-Ulam and Lusternik-Schnirelmann.

## 6. Another Combinatorial Result

The following result depends on the parity of the dimension of the pseudomanifold. It can be proved in a manner similar to the proof of Theorem 1 given in Section 3. Instead of giving the proof in detail, we indicate only that, because of the additive property, it suffices to consider the case in which $M^n$ consists of a single $n$-simplex and its faces.

THEOREM 2. *Let $M^n$ be a coherently oriented n-pseudomanifold.*

*Let $\varphi$ be a function which assigns to each vertex $v$ of $M^n$ an arbitrary non-negative integer $\varphi(v)$. If $n$ is even, then*

$$\sum_{0 \leq k_0 < k_1 < \cdots < k_n} \alpha(k_0, k_1, k_2, ..., k_n) = \sum_{0 \leq k_0 < k_1 < \cdots < k_{n-1}} \beta(k_0, k_1, k_2, ..., k_{n-1}). \tag{18}$$

*If $n$ is odd, then*

$$\sum_{0 < k_1 < k_2 < \cdots < k_n} \alpha(0, k_1, k_2, ..., k_n) = \sum_{0 < k_0 < k_1 < \cdots < k_{n-1}} \beta(k_0, k_1, k_2, ..., k_{n-1}). \tag{19}$$

Notice that the oriented version of Sperner's lemma can also be derived from Theorem 2.

## References


1. S. S. Cairns, *Introductory Topology*, Ronald Press, New York, 1961.
2. D. I. A. Cohen, On the Sperner Lemma *J. Comb. Theory* **2** (1967), 585–587 (preceding article).
3. K. Fan, A Generalization of Tucker's Combinatorial Lemma with Topological Applications, *Ann. of Math.* **56** (1952), 431–437.
4. K. Fan, Combinatorial Properties of Certain Simplicial and Cubical Vertex Maps, *Arch. Math.* **11** (1960), 368–377.
5. M. A. Krasnosel'skii and S. G. Krein, On a Proof of a Theorem on the Category of a Projective Space (in Russian), *Ukrain. Mat. Zh.* **1** (1949), 99–102.
6. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford-London, 1964.
7. A. W. Tucker, Some Topological Properties of Disk and Sphere, *Proc. First Canadian Math. Congress*, (1945), 285–309.